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The novel conformable methods to solve conformable time- fractional coupled Whitham-Broer-Kaup equation

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Abstract

This study employs two innovative techniques, namely the conformable q-homotopy analysis transform method (Cq-HATM) and the conformable Elzaki Adomian decomposition method (CEADM), to investigate the numerical solutions for the conformable time-fractional coupled Whitham-Broer-Kaup problem. One of the two novel approaches offered is Cq-HATM, which is a hybrid approach that integrates the q-homotopy analysis transform method with Laplace transform using the concept of conformable derivative. The CEADM method, like the aforementioned approach, is a hybrid method that integrates the Adomian decomposition method and Elzaki transform by utilizing the concept of conformable derivative. The computer simulations were conducted in order to provide empirical evidence supporting the efficacy and reliability of the recommended methodologies. Upon comparing the precise solutions with the obtained solutions, it becomes evident that both of the novel methodologies exhibit simplicity, efficacy, and proficiency in addressing nonlinear conformable time-fractional partial differential equations.

Keywords: Conformable time-fractional coupled Whitham-Broer-Kaup equation; Conformable q-homotopy analysis transform method; Conformable Elzaki transform

Mathematics Subject Classification: 35R11; 35A20; 35C05

1 Introduction

The topic of fractional calculus has been extensively studied and defined by numerous distinguished scientists. The researchers have formulated novel conceptualizations of fractional calculus, which have subsequently formed the fundamental framework for the field of fractional analysis. Fractional partial differential equations are frequently employed in the formulation of nonlinear models and the examination of dynamical systems. Fractional calculus has been utilized to analyze and study several subjects, such as chaos theory, financial models, disordered environments, and optics. The determination of nonlinear issues in nature relies heavily on the solutions of fractional differential equations. Due to the inherent complexity of fractional differential equations that capture nonlinear events,

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a diverse range of analytical and numerical methodologies are employed in order to get precise solutions [1–17].

In a recent publication, Khalil et al. have introduced a novel conceptualization of fractional derivative and fractional integral. The authors demonstrated that the newly proposed definition exhibits the characteristic qualities of the classical derivative as defined in classical analysis, while also employing a limit form similar to the classical derivative definition. In his work, the author introduces a novel definition of the fractional derivative. This definition incorporates various mathematical principles, including the product rule, quotient rule, chain rule, fractional Rolle's theorem, and fractional mean value theorems. The utilization of the conformable fractional derivative is regarded as a fundamental and very advantageous technique. Furthermore, it facilitates our ability to articulate the conduct exhibited by tangible entities. The utilization of the conformable fractional derivative serves as an innovative approach for addressing intricate problem sets. Fractional order models are commonly employed in the field of engineering and applied sciences due to their ability to offer a more precise representation of real-world phenomena. Conformable fractional derivatives have been employed by numerous scholars across various academic subjects. The utilization of the conformable fractional operator serves as a means to overcome some limitations present in existing fractional operators. The set of concepts encompassed in this topic comprises the mean value theorem, the chain rule, the product rule for differentiating two functions, the derivative of the quotient of two functions, and Rolle's theorem [18].

The Caputo derivative is more conventional, capturing non-local memory effects, and is well-suited for modeling real-world systems. The conformable derivative offers a simpler and more intuitive approach, with properties closer to classical derivatives, making it useful for specific applications where a direct generalization is desired without the complexities of integral definitions [1, 18].

The definition of the conformable derivative is simple and easy to comprehend; it is defined in a manner that is similar to the classical first-order derivative. Many classical features, such the product rule, chain rule, and quotient rule, are retained in the conformable derivative but are frequently changed or eliminated in other fractional derivatives like Caputo. Because of this preservation, conformable derivatives are easier to work with and calculate using standard calculus methods. The conformable derivative places more emphasis on local behavior at the point of differentiation than the Caputo derivative, which takes into account a function's whole historical behavior. In models where global memory effects are either superfluous or not as noticeable, this localization may be useful. Since the conformable derivative does not involve the intricate integral structures that are present in other fractional derivatives, it requires less computing power. This makes it appropriate for systems that need to perform rapid computations, such signal processing and control theory, and for real-time applications. Due to their simple application of beginning conditions, conformable derivatives make it relatively easier to formulate and solve initial value problems. Compared to other fractional derivatives that would need certain fractional beginning conditions, this can make issue formulation easier. Differential equation setup and solving are made simpler by conformable derivatives, particularly when using the laborious classical fractional calculus procedures. They provide a reasonable compromise between tractability and capturing fractional behavior. Owing to its ease of use, the conformable derivative has found application in a number of domains, including biology

modeling, dynamical systems, control theory, and economics, where a more direct method of fractional differentiation is advantageous [1, 18].

The Elzaki transform method (ETM), first introduced by Elzaki, has been utilized for solving linear ordinary differential equations with constant coefficients. Elzaki employed the differential transform approach in conjunction with the Elzaki transform (ET) to address a number of nonlinear differential equations. The Homotopy Perturbation Elzaki Transform Method (HPETM) was initially introduced by Elzaki and Hilal. Furthermore, the HPETM has successfully solved three nonlinear partial differential equations (PDEs). Elzaki and Kim employed a novel hybrid approach that combines the Elzaki transform (ET) with the modified variational iteration method to solve the radial diffusivity and shock wave equations. Aggarwal et al. employed ET to obtain the solutions of linear Volterra integral equations of the first kind. Jena and Chakraverty employed the HPETM to obtain a solution for the system of time-fractional Navier-Stokes equations [19–24].

Integral transforms like the Elzaki and Laplace are frequently used in physics, engineering, and mathematics to investigate systems and solve differential equations. The following are the main variations between them: The Laplace transform is particularly helpful in signal processing, control systems, and the solution of ordinary differential equations. Functions with a variety of boundary conditions can be handled using the ET, especially those that are difficult for Laplace transforms to handle. Compared to Laplace, it is somewhat more recent and less widely utilized. There are numerous tables and methods available for the well-defined and well-researched inversion of Laplace transform. The ET inversion is a little trickier to apply because it is less well-established, less researched, and has fewer tables available. Since the Laplace transform readily adjusts to different definitions of derivatives, it is a useful tool for managing conformable fractional calculus. This is a very new field of study since conformable calculus adaptations for Elzaki transform are still being worked out. Because of its familiarity and the ease with which fractional calculus techniques may be used, the Laplace transform is frequently utilized in fractional differential equations. Alternative methods for dealing with conformable fractional derivatives are provided by the Elzaki transform, albeit it is less well-known and documented [19–21].

However, it is important to note that the fractional order has the potential to exhibit variations both time and space. This particular issue pertains to a rapidly advancing field of fractional partial differential equations (FPDEs) that involve variable order fractional operators. Numerous robust numerical methodologies have been devised and documented in scholarly literature, with notable contributions from esteemed academics in the domain. Several methods have been proposed in the literature for solving mathematical problems. These methods include the Adomian Decomposition Method (ADM), the Homotopy Analysis Method (HAM), the Homotopy Perturbation Method (HPM), the Collocation Method, the Sumudu Transform Method (STM), the Differential Transformation Method (DTM), and the Variational Iteration Method (VIM) [25–32].

The ability of q -HATM to explicitly control convergence using auxiliary parameters is one of its main features. In order to continuously distort the challenging original issue into a more manageable form that is easier to solve analytically or numerically, the approach introduces the idea of embedding parameters, or q . A frequent problem in series solutions, divergence is greatly decreased by this feature. The use of transform techniques, such as Laplace transform, within the homotopy analysis framework facilitates the efficient treatment of initial value problems and boundary conditions. One important method for re-

solving difficult problems, such as nonlinear differential equations and integral equations, is the ADM. Its significance stems from a few essential elements. Unlike traditional techniques, which frequently call for discretization, linearization, or perturbation, ADM excels in solving nonlinear equations. Because it can handle a large range of problems directly in their original nonlinear forms, this makes ADM especially strong. The application of Adomian polynomials, which methodically break down nonlinear variables into a number of more manageable parts, is a key component of ADM. It is possible to handle nonlinearities well with this decomposition, which facilitates the construction of approximations without sacrificing the intrinsic complexity of the problem. The process builds a solution iteratively, beginning with a preliminary estimate and refining it over time. In many situations, quick convergence results from iterations that build upon one other. This comes in especially handy when alternative approaches can have trouble achieving a rapid convergence or necessitate intricate calculations. ADM manages differential equations without the requirement for certain transformations, like Fourier or Laplace transforms, unlike other approaches. Due to its simplicity, it may be applied to a variety of problems and is easily accessed, particularly in situations when the solution process could be complicated by these transforms. By include more terms in the series, ADM offers precise approximations that are easily refined. When simulating intricate physical processes, analysts can directly influence the precision and accuracy of the result. Because ADM can efficiently solve ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, and stochastic systems, it is employed in a wide range of domains, including physics, engineering, biology, and economics. Its wide range of applications emphasizes its significance and adaptability. In conclusion, the ADM is significant because it provides a methodical, effective, and extremely flexible way to tackle nonlinear issues that are frequently challenging to handle with traditional techniques [31, 33].

Shallow water waves are described by a system of nonlinear partial differential equations known as the Whitham-Broer-Kaup (WBK) equations. Wave interaction is represented by the nonlinear terms, which enable the creation of structures like shock waves and solitons. Waves of different wavelengths travel at varying speeds due to dispersive effects, which are modeled by the third-order derivative part in the second equation. The WBK equations are used in physics, coastal and ocean engineering, and other disciplines where the dynamics of shallow water waves are crucial. In order to comprehend wave dynamics in diverse physical contexts, researchers frequently examine the unique solutions of the WBK equations, such as traveling wave solutions, periodic wave solutions, and solitary wave solutions [33].

In this paper, we examine the conformable time-fractional coupled Whitham-Broer-Kaup equation system (CTFCWBKES) [33]

$$\begin{cases} {}_t T_\alpha u(x, t) + u(x, t) \frac{\partial u(x, t)}{\partial x} + \frac{1}{2} \frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial w(x, t)}{\partial x} = 0, \\ {}_t T_\alpha w(x, t) + w(x, t) \frac{\partial u(x, t)}{\partial x} + u(x, t) \frac{\partial w(x, t)}{\partial x} - \frac{1}{2} \frac{\partial^2 w(x, t)}{\partial x^2} = 0, \\ 0 < \alpha \leq 1, 0 < t \leq 1, -100 \leq x \leq 100, \end{cases} \quad (1)$$

The main aim of this study is to present the new two techniques: The first is Cq-HATM, which is a combination of the conformable Laplace transform and the q-homotopy analysis transform method. The second is CEADM, which is a combination of the conformable

ET and the Adomian decomposition method. These techniques numerically investigate all problems with initial values.

The second is CEADM, which is a combination of the conformable Elzaki transform and the Adomian decomposition method. The primary objective of this study is to acquire innovative numerical solutions for the conformable time-fractional coupled Whitham-Broer-Kaup equation by the utilization of the conformable q -homotopy analysis transform method (Cq-HATM). The second objective of this study is to acquire innovative numerical solutions for the conformable time-fractional coupled Whitham-Broer-Kaup equation by the utilization of the conformable Elzaki Adomian decomposition method (CEADM).

Presented below is an enumeration of the remaining components of the investigation. The second section of the text provides an explanation of the fundamental principles underlying conformable fractional calculus and the Elzaki transform. The following section introduces the novel conformable fractional numerical techniques. Section 4 presents an illustrative instance of the conformable time-fractional coupled Whitham-Broer-Kaup equation. The outcome is presented in Sect. 5.

2 Main definitions and theorems

This section provides some fundamental definitions.

Definition 2.1 [18, 34–36] Let a function $g : [0, \infty) \rightarrow \mathbb{R}$. Then, the conformable fractional derivative of g order α is defined as

$$T_{\alpha}(g)(x) = \lim_{\varepsilon \rightarrow 0} \frac{g(x + \varepsilon x^{1-\alpha}) - g(x)}{\varepsilon}, \quad \alpha \in (0, 1]. \quad (2)$$

for all $x > 0$.

Theorem 2.1 [18, 34–36] Let $\alpha \in (0, 1]$ and g, h be α -differentiable at a point $x > 0$. Then

$$(i) \quad T_{\alpha}(ag + bh) = aT_{\alpha}(g) + bT_{\alpha}(h), \quad \text{for all } a, b \in \mathbb{R}, \quad (3)$$

$$(ii) \quad T_{\alpha}(x^p) = px^{p-1}, \quad \text{for all } p \in \mathbb{R}, \quad (4)$$

$$(iii) \quad T_{\alpha}(\lambda) = 0, \quad \text{for all constant functions, } f(t) = \lambda, \quad (5)$$

$$(iv) \quad T_{\alpha}(gh) = gT_{\alpha}(h) + hT_{\alpha}(g), \quad (6)$$

$$(v) \quad T_{\alpha}\left(\frac{g}{h}\right) = \frac{hT_{\alpha}(g) - gT_{\alpha}(h)}{h^2}. \quad (7)$$

Definition 2.2 [37] Let $0 < \alpha \leq 1$, $g : [0, \infty) \rightarrow \mathbb{R}$ be function. Then, the conformable fractional Elzaki transform (CFET) of order α of g is described as

$${}_cE_{\alpha}[g(t)] = T_{\alpha}(v) = \int_0^{\infty} pK_{\alpha}(-p, t)g(t)d_{\alpha}t, \quad (8)$$

where $K_{\alpha}(-p, t) = E_{\alpha}\left(-\frac{1}{p}, t\right)$, $p > 0$.

Definition 2.3 [37] Let $0 < \alpha \leq 1$, $g : [0, \infty) \rightarrow \mathbb{R}$ be real function. The CFET for the conformable fractional derivative of the function $g(t)$ is defined as

$${}_cE_{\alpha}[T_{\alpha}g(t)](p) = \frac{1}{p} {}_cE_{\alpha}[g(t)](p) - pg(0). \quad (9)$$

3 The novel numerical techniques

The part provides an introduction to Cq-HATM and CEADM.

3.1 Conformable q-homotopy analysis transform method

A new method is presented. Consider the conformable time-fractional order nonlinear partial differential equation (CTFNPDE) to give the main idea of Cq-HATM:

$${}_t T_\alpha u(x, t) + Au(x, t) + Hu(x, t) = h(x, t), \quad t > 0, \quad n-1 < \alpha \leq n, \quad (10)$$

where A is a linear operator, H is a nonlinear operator, $h(x, t)$ is a source term, and ${}_t T_\alpha$ is a conformable fractional derivative of order α .

Now, by performing conformable Laplace transform (CLT) on Eq. (10) and using initial condition, then we get

$$s\mathcal{L}_\alpha[u(x, t)] - \sum_{m=0}^{k-1} u(x, 0) + \mathcal{L}_\alpha[Au(x, t)] + \mathcal{L}_\alpha[Hu(x, t)] = \mathcal{L}_\alpha[h(x, t)]. \quad (11)$$

If we simplify the Eq. (11), then we have

$$\mathcal{L}_\alpha[u(x, t)] - \frac{1}{s}u(x, 0) + \frac{1}{s}\mathcal{L}_\alpha[Au(x, t)] + \frac{1}{s}\mathcal{L}_\alpha[Hu(x, t)] - \frac{1}{s}\mathcal{L}_\alpha[h(x, t)] = 0. \quad (12)$$

We define the nonlinear operator by the assist of HAM for real function $\varphi(x, t; q)$ as follows

$$\begin{aligned} N[\varphi(x, t; q)] &= \mathcal{L}_\alpha[\varphi(x, t; q)] - \frac{1}{s}\varphi(x, t; q)(0^+) + \frac{1}{s}(\mathcal{L}_\alpha[A\varphi(x, t; q)] \\ &\quad + \mathcal{L}_\alpha[H\varphi(x, t; q)] - \mathcal{L}_\alpha[h(x, t)]), \end{aligned} \quad (13)$$

where $q \in [0, \frac{1}{n}]$.

We establish a homotopy in the following:

$$(1 - nq)\mathcal{L}_\alpha[\varphi(x, t; q) - u_0(x, t)] = hqH^+(x, t)H[\varphi(x, t; q)], \quad (14)$$

where, $h \neq 0$ is an auxiliary parameter and $\mathcal{L}_\alpha$ represents conformable Laplace transform. For $q = 0$ and $q = \frac{1}{n}$, the results in Eq. (14) are respectively provided:

$$\varphi(x, t; 0) = u_0(x, t), \quad \varphi\left(x, t; \frac{1}{n}\right) = u(x, t). \quad (15)$$

Therefore, by amplifying q from 0 to $\frac{1}{n}$, then the solution $\varphi(x, t; q)$ converges from $u_0(x, t)$ to the solution $u(x, t)$. Employing the Taylor theorem around q and expanding $\varphi(x, t; q)$ and then, we obtain

$$\varphi(x, t; q) = u_0(x, t) + \sum_{i=1}^{\infty} u_m(x, t)q^m, \quad (16)$$

where

$$u_m(x, t) = \frac{1}{m!} \frac{\partial^m \varphi(x, t; q)}{\partial q^m} \Big|_{q=0}. \quad (17)$$

Equation (16) converges at $q = \frac{1}{n}$ for the appropriate $w_0(x, t)$, n and h . Then, we have

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \left(\frac{1}{n}\right)^m. \quad (18)$$

If we differentiate the zeroth order deformation Eq. (14) m -times with respect to q and we divide by $m!$, respectively, then for $q = 0$, we obtain

$$\mathcal{L}_\alpha [u_m(x, t) - k_m u_{m-1}(x, t)] = h H^+(x, t) \mathcal{R}_m(\vec{u}_{m-1}), \quad (19)$$

where the vectors are defined by

$$\vec{u}_m = \{u_0(x, t), u_1(x, t), \dots, u_m(x, t)\}. \quad (20)$$

When we apply to the inverse CLT to Eq. (19), then we obtain

$$u_m(x, t) = k_m u_{m-1}(x, t) + h \mathcal{L}_\alpha^{-1} \left[H^+(x, t) \mathcal{R}_m(\vec{u}_{m-1}) \right], \quad (21)$$

where

$$\begin{aligned} \mathcal{R}_m(\vec{u}_{m-1}) &= \mathcal{L}_\alpha [u_{m-1}(x, t)] - \left(1 - \frac{k_m}{n}\right) \frac{1}{s} u_0(x, t) + \frac{1}{s} \mathcal{L}_\alpha (A u_{m-1}(x, t) \\ &\quad + H_{m-1}^+(x, t) - h(x, t)), \end{aligned} \quad (22)$$

and

$$k_m = \begin{cases} 0, & m \leq 1, \\ n, & m > 1. \end{cases} \quad (23)$$

where, H_m^+ is homotopy polynomial and presented as

$$H_m^+ = \frac{1}{m!} \frac{\partial^m \varphi(x, t; q)}{\partial q^m} \Big|_{q=0} \quad \text{and} \quad \varphi(x, t; q) = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \dots. \quad (24)$$

By utilizing Eqs. (21)–(22), then we obtain

$$\begin{aligned} u_m(x, t) &= (k_m + h) u_{m-1}(x, t) - \left(1 - \frac{k_m}{n}\right) \frac{1}{s} u_0(x, t) \\ &\quad + h \mathcal{L}_\alpha^{-1} \left[\left(\frac{1}{s} \mathcal{L}_\alpha [R u_{m-1}(x, t) + H_{m-1}(x, t) - f(x, t)] \right) \right]. \end{aligned} \quad (25)$$

By using Cq-HATM, the series solution is

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t) \left(\frac{1}{n}\right)^m. \quad (26)$$

3.2 Conformable Elzaki Adomian decomposition method

The examination of CTFNPDE in Eq. (10) is conducted:

Now, by performing conformable Elzaki transform (CET) on Eq. (10) and using initial condition, then we have

$$\frac{1}{\nu} {}_cE_\alpha [u(x, t)] - \nu u(x, 0) + {}_cE_\alpha [Au(x, t) + Hu(x, t)] = {}_cE_\alpha [h(x, t)]. \quad (27)$$

If we simplify the Eq. (27), then we get

$${}_cE_\alpha [u(x, t)] = \nu^2 u(x, 0) + \nu {}_cE_\alpha [h(x, t)] - \nu {}_cE_\alpha [Au(x, t) + Hu(x, t)]. \quad (28)$$

On applying inverse CET to Eq. (28), then we have

$$u(x, t) = C(x, t) - ({}_cE_\alpha)^{-1} \{ \nu {}_cE_\alpha [Au(x, t) + Hu(x, t)] \}, \quad (29)$$

where $C(x, t)$ is obtained from initial condition and non-homogeneous term. Now, assume that, the infinite series solution is of the form:

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t). \quad (30)$$

By employing Eqs. (29) and (30), then we have

$$\sum_{m=0}^{\infty} u_m(x, t) = C(x, t) - ({}_cE_\alpha)^{-1} \left\{ \nu {}_cE_\alpha \left[A \sum_{m=0}^{\infty} u_m(x, t) + \sum_{m=0}^{\infty} B_m(u_m(x, t)) \right] \right\}, \quad (31)$$

where $B_m(u_m)$ is Adomian polynomial and that denotes the nonlinear term $Hu(x, t)$. By comparing both of sides of Eq. (31), we have

$$u_0(x, t) = C(x, t), \quad (32)$$

$$u_1(x, t) = -({}_cE_\alpha)^{-1} \{ \nu {}_cE_\alpha [u_0(x, t) + B_0] \}, \quad (33)$$

$$u_2(x, t) = -({}_cE_\alpha)^{-1} \{ \nu {}_cE_\alpha [u_1(x, t) + B_1] \}, \quad (34)$$

$\vdots$

In a similar manner, the general recursive relation is derived by

$$u_{m+1}(x, t) = -({}_cE_\alpha)^{-1} \{ \nu {}_cE_\alpha [u_m(x, t) + B_m] \}, \quad m \geq 1, \quad (35)$$

Ultimately, the solution $u(x, t)$ is approximated as follows.

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t). \quad (36)$$

4 Application

The part aims to present visual representations of the time-fractional coupled Whitham-Broer-Kaup system in a conformable sense.

Example 4.1 [33] Consider the conformable time-fractional coupled Whitham-Broer-Kaup equation system (CTFCWBKES)

$$\begin{cases} {}_t T_\alpha u(x, t) + u(x, t) \frac{\partial u(x, t)}{\partial x} + \frac{1}{2} \frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial w(x, t)}{\partial x} = 0, \\ {}_t T_\alpha w(x, t) + w(x, t) \frac{\partial u(x, t)}{\partial x} + u(x, t) \frac{\partial w(x, t)}{\partial x} - \frac{1}{2} \frac{\partial^2 w(x, t)}{\partial x^2} = 0, \\ 0 < \alpha \leq 1, 0 < t \leq 1, -100 \leq x \leq 100, \end{cases} \quad (37)$$

with the initial conditions

$$\begin{aligned} u(x, 0) &= \xi - \kappa \coth[\kappa(x + \theta)], \\ w(x, 0) &= -\kappa^2 \operatorname{cosech}^2[\kappa(x + \theta)] \end{aligned} \quad (38)$$

Case (i) Cq-HATM solution

CLT is employed to Eq. (37), and by applying Eq. (38), the resulting expression is produced as

$$\begin{aligned} \mathcal{L}_\alpha [u(x, t)] - \frac{1}{s} u(x, 0) + \frac{1}{s} \mathcal{L}_\alpha \left[u \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \right] &= 0, \\ \mathcal{L}_\alpha [w(x, t)] - \frac{1}{s} w(x, 0) + \frac{1}{s} \mathcal{L}_\alpha \left[w \frac{\partial u}{\partial x} + u \frac{\partial w}{\partial x} - \frac{1}{2} \frac{\partial^2 w}{\partial x^2} \right] &= 0. \end{aligned} \quad (39)$$

The nonlinear operators are defined by employing Eq. (39):

$$\begin{aligned} N^1 [\varphi(x, t; q), \psi(x, t; q)] &= \mathcal{L}_\alpha [\varphi(x, t; q)] - \frac{1}{s} (\xi - \kappa \coth[\kappa(x + \theta)]) \\ &\quad + \frac{1}{s} \mathcal{L}_\alpha \left[\varphi(x, t; q) \frac{\partial \varphi(x, t; q)}{\partial x} \right. \\ &\quad \left. + \frac{1}{2} \frac{\partial^2 \varphi(x, t; q)}{\partial x^2} + \frac{\partial \psi(x, t; q)}{\partial x} \right], \end{aligned} \quad (40)$$

$$\begin{aligned} N^2 [\varphi(x, t; q), \psi(x, t; q)] &= \mathcal{L}_\alpha [\psi(x, t; q)] - \frac{1}{s} (-\kappa^2 \operatorname{cosech}^2[\kappa(x + \theta)]) \\ &\quad + \frac{1}{s} \mathcal{L}_\alpha \left[\varphi(x, t; q) \frac{\partial \psi(x, t; q)}{\partial x} \right. \\ &\quad \left. + \psi(x, t; q) \frac{\partial \varphi(x, t; q)}{\partial x} - \frac{1}{2} \frac{\partial^2 \psi(x, t; q)}{\partial x^2} \right]. \end{aligned} \quad (41)$$

The m -th order deformation equations are defined by the application of the proposed algorithm:

$$\mathcal{L}_\alpha [u_m(x, t) - k_m u_{m-1}(x, t)] = h \mathcal{R}_{1,m} [\vec{u}_{m-1}, \vec{w}_{m-1}], \quad (42)$$

$$\mathcal{L}_\alpha [w_m(x, t) - k_m w_{m-1}(x, t)] = h \mathcal{R}_{2,m} [\vec{u}_{m-1}, \vec{w}_{m-1}], \quad (43)$$

where

$$\begin{aligned} \mathcal{R}_{1,m} \left[\vec{u}_{m-1}, \vec{w}_{m-1} \right] &= \mathcal{L}_\alpha \left[\vec{u}_{m-1}(x, t) \right] - \left(1 - \frac{k_m}{n} \right) \frac{1}{s} (\xi - \kappa \coth [\kappa (x + \theta)]) \\ &\quad + \frac{1}{s} \mathcal{L}_\alpha \left[\sum_{r=0}^{m-1} u_r \frac{\partial u_{m-1-r}}{\partial x} + \frac{1}{2} \frac{\partial^2 u_{m-1}}{\partial x^2} + \frac{\partial w_{m-1}}{\partial x} \right], \end{aligned} \quad (44)$$

$$\begin{aligned} \mathcal{R}_{2,m} \left[\vec{u}_{m-1}, \vec{w}_{m-1} \right] &= \mathcal{L}_\alpha \left[\vec{w}_{m-1}(x, t) \right] - \left(1 - \frac{k_m}{n} \right) \frac{1}{s} (-\kappa^2 \operatorname{cosech}^2 [\kappa (x + \theta)]) \\ &\quad + \frac{1}{s} \mathcal{L}_\alpha \left[\sum_{r=0}^{m-1} u_r \frac{\partial w_{m-1-r}}{\partial x} + \sum_{r=0}^{m-1} w_r \frac{\partial u_{m-1-r}}{\partial x} - \frac{1}{2} \frac{\partial^2 w_{m-1}}{\partial x^2} \right]. \end{aligned} \quad (45)$$

By utilizing the inverse CLT to Eqs. (42)–(43), we obtain

$$u_m(x, t) = k_m u_{m-1}(x, t) + h \mathcal{L}_\alpha^{-1} \left\{ \mathcal{R}_{1,m} \left[\vec{u}_{m-1}, \vec{w}_{m-1} \right] \right\}, \quad (46)$$

$$w_m(x, t) = k_m w_{m-1}(x, t) + h \mathcal{L}_\alpha^{-1} \left\{ \mathcal{R}_{2,m} \left[\vec{u}_{m-1}, \vec{w}_{m-1} \right] \right\}. \quad (47)$$

By employing initial conditions, we are able to drive

$$u_0(x, t) = \xi - \kappa \coth [\kappa (x + \theta)], \quad (48)$$

$$w_0(x, t) = -\kappa^2 \operatorname{cosech}^2 [\kappa (x + \theta)]. \quad (49)$$

To get the values of $u_1(x, t)$ and $w_1(x, t)$, we substitute $m = 1$ into Eqs. (46)–(47), resulting in the following expressions:

$$u_1(x, t) = \frac{h \xi \kappa^2 t^\alpha}{\alpha \sinh^2 [\kappa (x + \theta)]}, \quad (50)$$

$$w_1(x, t) = \frac{2h \xi \kappa^3 \cosh [\kappa (x + \theta)] t^\alpha}{\alpha \sinh^3 [\kappa (x + \theta)]}. \quad (51)$$

In a similar vein, by substituting $m = 2$ into Eqs. (46)–(47), the resulting values for $u_2(x, t)$ and $w_2(x, t)$ can be obtained:

$$u_2(x, t) = \frac{(n + h) h \xi \kappa^2 t^\alpha}{\alpha \sinh^2 [\kappa (x + \theta)]} - \frac{h^2 \xi^2 \kappa^3 \cosh [\kappa (x + \theta)] t^{2\alpha}}{\alpha^2 \sinh^3 [\kappa (x + \theta)]}, \quad (52)$$

$$\begin{aligned} w_2(x, t) &= \frac{2(n + h) h \xi \kappa^3 \cosh [\kappa (x + \theta)] t^\alpha}{\alpha \sinh^3 [\kappa (x + \theta)]} \\ &\quad - \frac{2h^2 \xi^2 \kappa^4 (\cosh^2 [\kappa (x + \theta)] + \frac{1}{2}) t^{2\alpha}}{\alpha^2 \sinh^4 [\kappa (x + \theta)]}. \end{aligned} \quad (53)$$

By employing this approach, it is possible to identify the remaining terms. The solutions of the CTFCWBKES are determined through the Cq-HATM:

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \left(\frac{1}{n} \right)^m, \quad (54)$$

$$w(x, t) = w_0(x, t) + \sum_{m=1}^{\infty} w_m(x, t) \left(\frac{1}{n}\right)^m. \quad (55)$$

By substituting $\alpha = 1$, $n = 1$, $h = -1$ into Eqs. (54)–(55), we have that the resulting outcomes, denoted as $\sum_{m=1}^M u_m(x, t) \left(\frac{1}{n}\right)^m$ and $\sum_{m=1}^M w_m(x, t) \left(\frac{1}{n}\right)^m$, respectively, converge to the exact solutions $u(x, t) = \xi - \kappa \coth[\kappa(x + \theta - \xi t)]$ and $w(x, t) = -\kappa^2 \operatorname{cosech}^2[\kappa(x + \theta - \xi t)]$ of the CTFCWBKES when $M \rightarrow \infty$.

Case (ii) CEADM solution

By employing the CET to Eq. (37) and utilizing Eq. (38), the resulting expression is obtained.

$$\frac{1}{\nu} {}_cE_{\alpha} \{u(x, t)\} - \nu u(x, 0) = - {}_cE_{\alpha} \left[u(x, t) \frac{\partial u(x, t)}{\partial x} + \frac{1}{2} \frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial w(x, t)}{\partial x} \right], \quad (56)$$

$$\begin{aligned} \frac{1}{\nu} {}_cE_{\alpha} \{w(x, t)\} - \nu w(x, 0) = & - {}_cE_{\alpha} \left[w(x, t) \frac{\partial u(x, t)}{\partial x} + u(x, t) \frac{\partial w(x, t)}{\partial x} \right. \\ & \left. - \frac{1}{2} \frac{\partial^2 w(x, t)}{\partial x^2} \right]. \end{aligned} \quad (57)$$

Rearranging Eqs. (56)–(57), then we obtain

$${}_cE_{\alpha} \{u(x, t)\} = \nu^2 u(x, 0) - \nu {}_cE_{\alpha} \left[u(x, t) \frac{\partial u(x, t)}{\partial x} + \frac{1}{2} \frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial w(x, t)}{\partial x} \right], \quad (58)$$

$$\begin{aligned} {}_cE_{\alpha} \{w(x, t)\} = & \nu^2 w(x, 0) - \nu {}_cE_{\alpha} \left[w(x, t) \frac{\partial u(x, t)}{\partial x} + u(x, t) \frac{\partial w(x, t)}{\partial x} \right. \\ & \left. - \frac{1}{2} \frac{\partial^2 w(x, t)}{\partial x^2} \right]. \end{aligned} \quad (59)$$

By utilizing the inverse CET on Eqs. (58)–(59), we are able to derive the following result:

$$\begin{aligned} u(x, t) = & \xi - \kappa \coth[\kappa(x + \theta)] - ({}_cE_{\alpha})^{-1} \left\{ \nu {}_cE_{\alpha} \left[u(x, t) \frac{\partial u(x, t)}{\partial x} + \frac{1}{2} \frac{\partial^2 u(x, t)}{\partial x^2} \right. \right. \\ & \left. \left. + \frac{\partial w(x, t)}{\partial x} \right] \right\}, \end{aligned} \quad (60)$$

$$\begin{aligned} w(x, t) = & -\kappa^2 \operatorname{cosech}^2[\kappa(x + \theta)] - ({}_cE_{\alpha})^{-1} \left\{ \nu {}_cE_{\alpha} \left[w(x, t) \frac{\partial u(x, t)}{\partial x} \right. \right. \\ & \left. \left. + u(x, t) \frac{\partial w(x, t)}{\partial x} - \frac{1}{2} \frac{\partial^2 w(x, t)}{\partial x^2} \right] \right\}, \end{aligned} \quad (61)$$

Let us consider the assumption that the answer to the infinite series can be expressed in the following form:

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t), \quad (62)$$

$$w(x, t) = \sum_{m=0}^{\infty} w_m(x, t). \quad (63)$$

Utilizing ADM, Now, if we rewrite Eqs. (62)–(63), then it is obtained as

$$\sum_{m=0}^{\infty} u_m(x, t) = \xi - \kappa \coth[\kappa(x + \theta)] - (E_{\alpha}^c)^{-1} \left\{ v E_{\alpha}^c \left[\sum_{m=0}^{\infty} A_m + \frac{1}{2} \frac{\partial^2 u_{m-1}(x, t)}{\partial x^2} + \frac{\partial w_{m-1}(x, t)}{\partial x} \right] \right\}, \quad (64)$$

$$\sum_{m=0}^{\infty} w_m(x, t) = -\kappa^2 \operatorname{cosech}^2[\kappa(x + \theta)] - E_{\alpha}^{-1} \left\{ v E_{\alpha} \left[\sum_{m=0}^{\infty} B_m + \sum_{m=0}^{\infty} C_m - \frac{1}{2} \frac{\partial^2 w_{m-1}(x, t)}{\partial x^2} \right] \right\}, \quad (65)$$

where A_m , B_m and C_m are Adomian polynomials of the form $u \frac{\partial u}{\partial x} = \sum_{m=0}^{\infty} A_m$, $w \frac{\partial u}{\partial x} = \sum_{m=0}^{\infty} B_m$ and $u \frac{\partial w}{\partial x} = \sum_{m=0}^{\infty} C_m$.

By taking both sides of Eqs. (64)–(65) and making use of the initial condition (38) and Eqs. (64)–(65), we simply obtain the following iterations:

$$u_0(x, t) = \xi - \kappa \coth[\kappa(x + \theta)], \quad (66)$$

$$w_0(x, t) = -\kappa^2 \operatorname{cosech}^2[\kappa(x + \theta)], \quad (67)$$

$$u_1(x, t) = -\frac{\xi \kappa^2 t^{\alpha}}{\alpha \sinh^2[\kappa(x + \theta)]}, \quad (68)$$

$$w_1(x, t) = -\frac{2\xi \kappa^3 \cosh[\kappa(x + \theta)] t^{\alpha}}{\alpha \sinh^3[\kappa(x + \theta)]}, \quad (69)$$

$$u_2(x, t) = -\frac{\xi^2 \kappa^3 \cosh[\kappa(x + \theta)] t^{2\alpha}}{\alpha^2 \sinh^3[\kappa(x + \theta)]}, \quad (70)$$

$$w_2(x, t) = -\frac{2\xi^2 \kappa^4 \left(\cosh^2[\kappa(x + \theta)] + \frac{1}{2} \right) t^{2\alpha}}{\alpha^2 \sinh^4[\kappa(x + \theta)]}, \quad (71)$$

By continuing in a similar manner, the outcomes for CTFCWBKES can be derived.

$$\begin{aligned} u(x, t) &= \sum_{m=0}^{\infty} u_m(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots \\ &= \xi - \kappa \coth[\kappa(x + \theta)] - \frac{\xi \kappa^2 t^{\alpha}}{\alpha \sinh^2[\kappa(x + \theta)]} - \frac{\xi^2 \kappa^3 \cosh[\kappa(x + \theta)] t^{2\alpha}}{\alpha^2 \sinh^3[\kappa(x + \theta)]}, \end{aligned} \quad (72)$$

$$\begin{aligned} w(x, t) &= \sum_{m=0}^{\infty} w_m(x, t) = w_0(x, t) + w_1(x, t) + w_2(x, t) + \cdots \\ &= -\kappa^2 \operatorname{cosech}^2[\kappa(x + \theta)] - \frac{2\xi \kappa^3 \cosh[\kappa(x + \theta)] t^{\alpha}}{\alpha \sinh^3[\kappa(x + \theta)]} \\ &\quad - \frac{2\xi^2 \kappa^4 \left(\cosh^2[\kappa(x + \theta)] + \frac{1}{2} \right) t^{2\alpha}}{\alpha^2 \sinh^4[\kappa(x + \theta)]}. \end{aligned} \quad (73)$$

Figure 1 displays the 3D graphical representations of Cq-HATM, the exact solution, and the absolute error for $u(x, t)$ and $w(x, t)$.

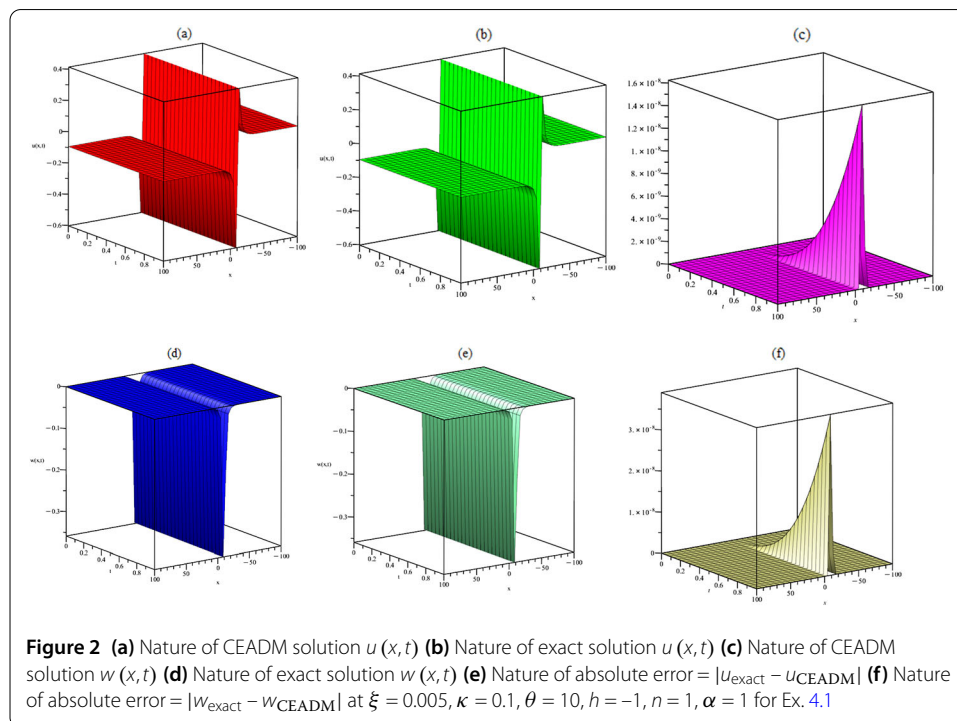
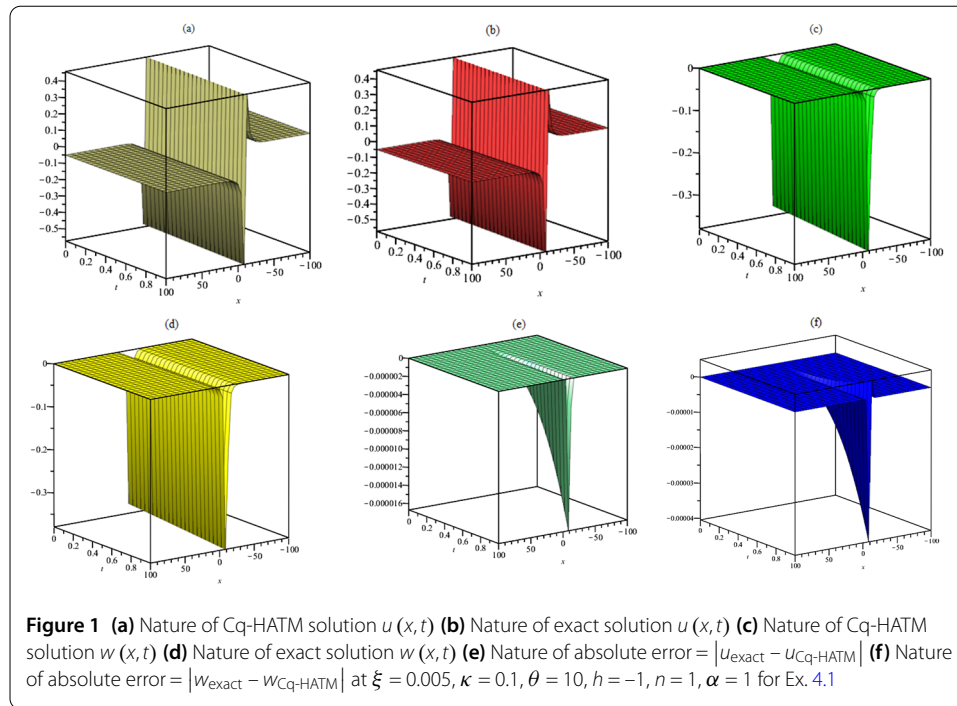


Figure 2 presents the three-dimensional graphical depictions of CEADM, the exact solution, and the absolute error for $u(x,t)$ and $w(x,t)$.

Figure 3 shows the two-dimensional graphical representations of Cq-HATM for $u(x,t)$, $w(x,t)$ solutions and the exact solution for different α values.

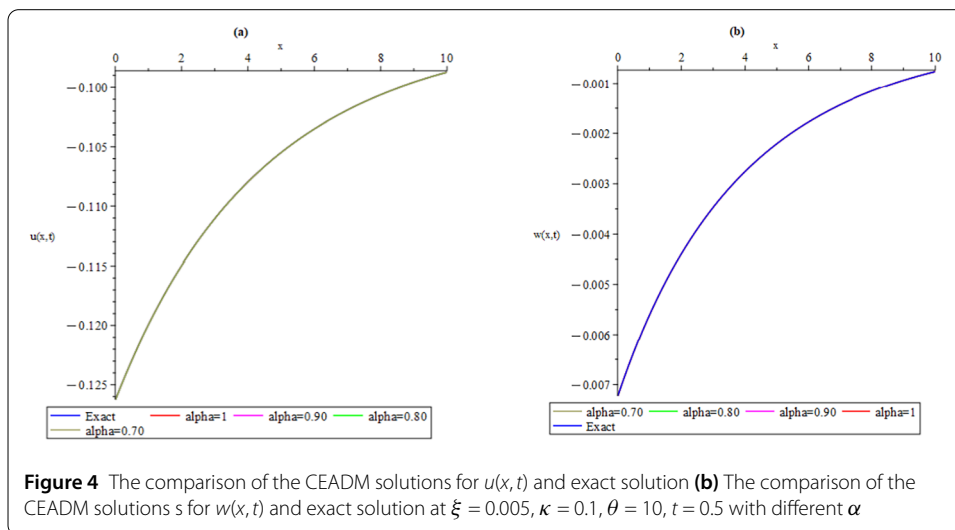
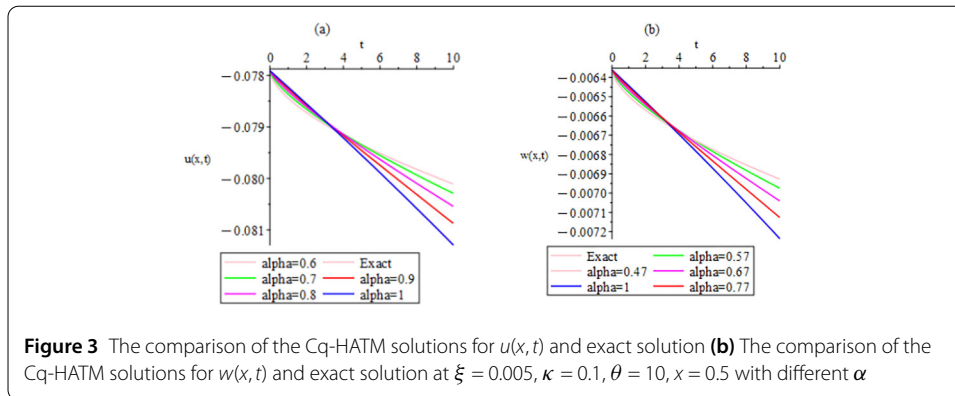


Figure 4 displays the graphical depictions of CEADM for the solutions $u(x,t)$ and $w(x,t)$, as well as the exact solution, in a two-dimensional format. These representations vary based on the different α values.

Table 1 shows the numerical solution of $u(x,t)$ obtained from the solution of CTFCWBKES with Cq-HATM for different x, t and α values.

Table 2 presents the numerical solution of the function $w(x,t)$, which was derived from the solution of the CTFCWBKES using the Cq-HATM. The table displays the results for various values of x, t , and α .

Table 3 demonstrates the numerical solution of $u(x,t)$ obtained from the solution of CTFCWBKES with CEADM for distinct x, t and α values.

Table 4 shows the numerical solution of the function $w(x,t)$, which was derived from the solution of the CTFCWBKES using the CEADM. The table displays the results for various values of x, t , and α .

5 Results and discussion

Figure 1 displays the three-dimensional graphs of the numerical solutions obtained by the Cq-HATM, as well as the exact solutions and the absolute errors between the Cq-HATM solutions and the exact solutions for the CTFCWBKES. Figure 2 displays the three-dimensional graphs depicting the numerical solutions obtained by the CEADM for

Table 1 Numerical solution of $u(x, t)$ by Cq-HATM for CTFCWBKES with different x , t and α at $\xi = 0.005$, $\kappa = 0.1$, $\theta = 10$, $h = -1$, $n = 1$

x	t	$\alpha = 0.47$	$\alpha = 0.57$	$\alpha = 0.67$	$\alpha = 0.77$	$\alpha = 1$
0.1	0.001	2.8×10^{-5}	1.1×10^{-5}	4.7×10^{-6}	1.8×10^{-6}	1.2×10^{-17}
	0.002	3.9×10^{-5}	1.7×10^{-5}	7.4×10^{-6}	3.1×10^{-6}	9.6×10^{-17}
	0.003	4.7×10^{-5}	2.1×10^{-5}	9.6×10^{-6}	4.1×10^{-6}	3.2×10^{-16}
	0.004	5.4×10^{-5}	2.5×10^{-5}	1.1×10^{-5}	5.1×10^{-6}	7.7×10^{-16}
	0.005	6.0×10^{-5}	2.8×10^{-5}	1.3×10^{-5}	5.9×10^{-6}	1.5×10^{-15}
0.2	0.001	2.8×10^{-5}	1.1×10^{-5}	4.6×10^{-6}	1.8×10^{-6}	1.1×10^{-17}
	0.002	3.8×10^{-5}	1.6×10^{-5}	7.2×10^{-6}	3.0×10^{-6}	9.3×10^{-17}
	0.003	4.6×10^{-5}	2.0×10^{-5}	9.4×10^{-6}	4.0×10^{-6}	3.1×10^{-16}
	0.004	5.3×10^{-5}	2.4×10^{-5}	1.1×10^{-5}	4.9×10^{-6}	7.4×10^{-16}
	0.005	5.8×10^{-5}	2.7×10^{-5}	1.3×10^{-5}	5.8×10^{-6}	1.4×10^{-15}
0.3	0.001	2.7×10^{-5}	1.1×10^{-5}	4.5×10^{-6}	1.7×10^{-6}	1.1×10^{-17}
	0.002	3.7×10^{-5}	1.6×10^{-5}	7.1×10^{-6}	2.9×10^{-6}	8.9×10^{-17}
	0.003	4.5×10^{-5}	2.0×10^{-5}	9.1×10^{-6}	3.9×10^{-6}	3.0×10^{-16}
	0.004	5.1×10^{-5}	2.3×10^{-5}	1.1×10^{-5}	4.8×10^{-6}	7.1×10^{-16}
	0.005	5.7×10^{-5}	2.7×10^{-5}	1.2×10^{-5}	5.6×10^{-6}	1.3×10^{-15}
0.4	0.001	2.6×10^{-5}	1.0×10^{-5}	4.4×10^{-6}	1.7×10^{-6}	1.0×10^{-17}
	0.002	3.6×10^{-5}	1.5×10^{-5}	6.9×10^{-6}	2.8×10^{-6}	8.6×10^{-17}
	0.003	4.4×10^{-5}	1.9×10^{-5}	8.9×10^{-6}	3.8×10^{-6}	2.9×10^{-16}
	0.004	5.0×10^{-5}	2.3×10^{-5}	1.0×10^{-5}	4.7×10^{-6}	6.8×10^{-16}
	0.005	5.5×10^{-5}	2.6×10^{-5}	1.2×10^{-5}	5.5×10^{-6}	1.3×10^{-15}
0.5	0.001	2.6×10^{-5}	1.0×10^{-5}	4.3×10^{-6}	1.7×10^{-6}	1.0×10^{-17}
	0.002	3.5×10^{-5}	1.5×10^{-5}	6.7×10^{-6}	2.8×10^{-6}	8.2×10^{-17}
	0.003	4.3×10^{-5}	1.9×10^{-5}	8.7×10^{-6}	3.7×10^{-6}	2.7×10^{-16}
	0.004	4.9×10^{-5}	2.2×10^{-5}	1.0×10^{-5}	4.6×10^{-6}	6.6×10^{-16}
	0.005	5.4×10^{-5}	2.5×10^{-5}	1.2×10^{-5}	5.3×10^{-6}	1.2×10^{-15}

Table 2 Numerical solution of $w(x, t)$ by Cq-HATM for CTFCWBKES with different x , t and α at $\xi = 0.005$, $\kappa = 0.1$, $\theta = 10$, $h = -1$, $n = 1$

x	t	$\alpha = 0.47$	$\alpha = 0.57$	$\alpha = 0.67$	$\alpha = 0.77$	$\alpha = 1$
0.1	0.001	7.5×10^{-6}	3.0×10^{-6}	1.2×10^{-6}	4.9×10^{-7}	4.7×10^{-18}
	0.002	1.0×10^{-5}	4.4×10^{-6}	1.9×10^{-6}	8.1×10^{-7}	3.8×10^{-17}
	0.003	1.2×10^{-5}	5.6×10^{-6}	2.5×10^{-6}	1.0×10^{-6}	1.2×10^{-16}
	0.004	1.4×10^{-5}	6.5×10^{-6}	3.0×10^{-6}	1.3×10^{-6}	3.0×10^{-16}
	0.005	1.5×10^{-5}	7.4×10^{-6}	3.4×10^{-6}	1.5×10^{-6}	5.9×10^{-16}
0.2	0.001	7.3×10^{-6}	2.9×10^{-6}	1.2×10^{-6}	4.7×10^{-7}	4.5×10^{-18}
	0.002	1.0×10^{-5}	4.3×10^{-6}	1.8×10^{-6}	7.8×10^{-7}	3.6×10^{-17}
	0.003	1.2×10^{-5}	5.4×10^{-6}	2.4×10^{-6}	1.0×10^{-6}	1.2×10^{-16}
	0.004	1.3×10^{-5}	6.3×10^{-6}	2.9×10^{-6}	1.2×10^{-6}	2.9×10^{-16}
	0.005	1.5×10^{-5}	7.2×10^{-6}	3.3×10^{-6}	1.5×10^{-6}	5.6×10^{-16}
0.3	0.001	7.0×10^{-6}	2.8×10^{-6}	1.1×10^{-6}	4.6×10^{-7}	4.3×10^{-18}
	0.002	9.7×10^{-6}	4.2×10^{-6}	1.8×10^{-6}	7.6×10^{-7}	3.4×10^{-17}
	0.003	1.1×10^{-5}	5.2×10^{-6}	2.3×10^{-6}	1.0×10^{-6}	1.1×10^{-16}
	0.004	1.3×10^{-5}	6.1×10^{-6}	2.8×10^{-6}	1.2×10^{-6}	2.7×10^{-16}
	0.005	1.4×10^{-5}	6.9×10^{-6}	3.2×10^{-6}	1.4×10^{-6}	5.4×10^{-16}
0.4	0.001	6.8×10^{-6}	2.7×10^{-6}	1.1×10^{-6}	4.4×10^{-7}	4.1×10^{-18}
	0.002	9.4×10^{-6}	4.0×10^{-6}	1.7×10^{-6}	7.4×10^{-7}	3.3×10^{-17}
	0.003	1.1×10^{-5}	5.1×10^{-6}	2.3×10^{-6}	9.9×10^{-7}	1.1×10^{-16}
	0.004	1.3×10^{-5}	5.9×10^{-6}	2.7×10^{-6}	1.2×10^{-6}	2.6×10^{-16}
	0.005	1.4×10^{-5}	6.7×10^{-6}	3.1×10^{-6}	1.4×10^{-6}	5.1×10^{-16}
0.5	0.001	6.6×10^{-6}	2.7×10^{-6}	1.1×10^{-6}	4.3×10^{-7}	3.9×10^{-18}
	0.002	9.1×10^{-6}	3.9×10^{-6}	1.7×10^{-6}	7.1×10^{-7}	3.1×10^{-17}
	0.003	1.1×10^{-5}	4.9×10^{-6}	2.2×10^{-6}	9.6×10^{-7}	1.0×10^{-16}
	0.004	1.2×10^{-5}	5.8×10^{-6}	2.6×10^{-6}	1.1×10^{-6}	2.5×10^{-16}
	0.005	1.3×10^{-5}	6.5×10^{-6}	3.0×10^{-6}	1.3×10^{-6}	4.9×10^{-16}

Table 3 Numerical solution of $u(x, t)$ by CEADM for CTFCWBKES with different x , t and α at $\xi = 0.005$, $\kappa = 0.1$, $\theta = 10$, $h = -1$, $n = 1$

x	t	$\alpha = 0.47$	$\alpha = 0.57$	$\alpha = 0.67$	$\alpha = 0.77$	$\alpha = 1$
0.1	0.001	2.8×10^{-5}	1.1×10^{-5}	4.7×10^{-6}	1.8×10^{-6}	1.2×10^{-17}
	0.002	3.9×10^{-5}	1.7×10^{-5}	7.4×10^{-6}	3.1×10^{-6}	9.6×10^{-17}
	0.003	4.7×10^{-5}	2.1×10^{-5}	9.6×10^{-6}	4.1×10^{-6}	3.2×10^{-16}
	0.004	5.4×10^{-5}	2.5×10^{-5}	1.1×10^{-5}	5.1×10^{-6}	7.7×10^{-16}
	0.005	6.0×10^{-5}	2.8×10^{-5}	1.3×10^{-5}	5.9×10^{-6}	1.5×10^{-15}
0.2	0.001	2.8×10^{-5}	1.1×10^{-5}	4.6×10^{-6}	1.8×10^{-6}	1.1×10^{-17}
	0.002	3.8×10^{-5}	1.6×10^{-5}	7.2×10^{-6}	3.0×10^{-6}	9.3×10^{-17}
	0.003	4.6×10^{-5}	2.0×10^{-5}	9.4×10^{-6}	4.0×10^{-6}	3.1×10^{-16}
	0.004	5.3×10^{-5}	2.4×10^{-5}	1.1×10^{-5}	4.9×10^{-6}	7.4×10^{-16}
	0.005	5.8×10^{-5}	2.7×10^{-5}	1.3×10^{-5}	5.8×10^{-6}	1.4×10^{-15}
0.3	0.001	2.7×10^{-5}	1.1×10^{-5}	4.5×10^{-6}	1.7×10^{-6}	1.1×10^{-17}
	0.002	3.7×10^{-5}	1.6×10^{-5}	7.1×10^{-6}	2.9×10^{-6}	8.9×10^{-17}
	0.003	4.5×10^{-5}	2.0×10^{-5}	9.1×10^{-6}	3.9×10^{-6}	3.0×10^{-16}
	0.004	5.1×10^{-5}	2.3×10^{-5}	1.1×10^{-5}	4.8×10^{-6}	7.1×10^{-16}
	0.005	5.7×10^{-5}	2.7×10^{-5}	1.2×10^{-5}	5.6×10^{-6}	1.3×10^{-15}
0.4	0.001	2.6×10^{-5}	1.0×10^{-5}	4.4×10^{-6}	1.7×10^{-6}	1.0×10^{-17}
	0.002	3.6×10^{-5}	1.5×10^{-5}	6.9×10^{-6}	2.8×10^{-6}	8.6×10^{-17}
	0.003	4.4×10^{-5}	1.9×10^{-5}	8.9×10^{-6}	3.8×10^{-6}	2.9×10^{-16}
	0.004	5.0×10^{-5}	2.3×10^{-5}	1.0×10^{-5}	4.7×10^{-6}	6.8×10^{-16}
	0.005	5.5×10^{-5}	2.6×10^{-5}	1.2×10^{-5}	5.5×10^{-6}	1.3×10^{-15}
0.5	0.001	2.6×10^{-5}	1.0×10^{-5}	4.3×10^{-6}	1.7×10^{-6}	1.0×10^{-17}
	0.002	3.5×10^{-5}	1.5×10^{-5}	6.7×10^{-6}	2.8×10^{-6}	8.2×10^{-17}
	0.003	4.3×10^{-5}	1.9×10^{-5}	8.7×10^{-6}	3.7×10^{-6}	2.7×10^{-16}
	0.004	4.9×10^{-5}	2.2×10^{-5}	1.0×10^{-5}	4.6×10^{-6}	6.6×10^{-16}
	0.005	5.4×10^{-5}	2.5×10^{-5}	1.2×10^{-5}	5.3×10^{-6}	1.2×10^{-15}

Table 4 Numerical solution of $w(x, t)$ by CEADM for CTFCWBKES with different x , t and α at $\xi = 0.005$, $\kappa = 0.1$, $\theta = 10$, $h = -1$, $n = 1$

x	t	$\alpha = 0.47$	$\alpha = 0.57$	$\alpha = 0.67$	$\alpha = 0.77$	$\alpha = 1$
0.1	0.001	7.5×10^{-6}	3.0×10^{-6}	1.2×10^{-6}	4.9×10^{-7}	4.7×10^{-18}
	0.002	1.0×10^{-5}	4.4×10^{-6}	1.9×10^{-6}	8.1×10^{-7}	3.8×10^{-17}
	0.003	1.2×10^{-5}	5.6×10^{-6}	2.5×10^{-6}	1.0×10^{-6}	1.2×10^{-16}
	0.004	1.4×10^{-5}	6.5×10^{-6}	3.0×10^{-6}	1.3×10^{-6}	3.0×10^{-16}
	0.005	1.5×10^{-5}	7.4×10^{-6}	3.4×10^{-6}	1.5×10^{-6}	5.9×10^{-16}
0.2	0.001	7.3×10^{-6}	2.9×10^{-6}	1.2×10^{-6}	4.7×10^{-7}	4.5×10^{-18}
	0.002	1.0×10^{-5}	4.3×10^{-6}	1.8×10^{-6}	7.8×10^{-7}	3.6×10^{-17}
	0.003	1.2×10^{-5}	5.4×10^{-6}	2.4×10^{-6}	1.0×10^{-6}	1.2×10^{-16}
	0.004	1.3×10^{-5}	6.3×10^{-6}	2.9×10^{-6}	1.2×10^{-6}	2.9×10^{-16}
	0.005	1.5×10^{-5}	7.2×10^{-6}	3.3×10^{-6}	1.5×10^{-6}	5.6×10^{-16}
0.3	0.001	7.0×10^{-6}	2.8×10^{-6}	1.1×10^{-6}	4.6×10^{-7}	4.3×10^{-18}
	0.002	9.7×10^{-6}	4.2×10^{-6}	1.8×10^{-6}	7.6×10^{-7}	3.4×10^{-17}
	0.003	1.1×10^{-5}	5.2×10^{-6}	2.3×10^{-6}	1.0×10^{-6}	1.1×10^{-16}
	0.004	1.3×10^{-5}	6.1×10^{-6}	2.8×10^{-6}	1.2×10^{-6}	2.7×10^{-16}
	0.005	1.4×10^{-5}	6.9×10^{-6}	3.2×10^{-6}	1.4×10^{-6}	5.4×10^{-16}
0.4	0.001	6.8×10^{-6}	2.7×10^{-6}	1.1×10^{-6}	4.4×10^{-7}	4.1×10^{-18}
	0.002	9.4×10^{-6}	4.0×10^{-6}	1.7×10^{-6}	7.4×10^{-7}	3.3×10^{-17}
	0.003	1.1×10^{-5}	5.1×10^{-6}	2.3×10^{-6}	9.9×10^{-7}	1.1×10^{-16}
	0.004	1.3×10^{-5}	5.9×10^{-6}	2.7×10^{-6}	1.2×10^{-6}	2.6×10^{-16}
	0.005	1.4×10^{-5}	6.7×10^{-6}	3.1×10^{-6}	1.4×10^{-6}	5.1×10^{-16}
0.5	0.001	6.6×10^{-6}	2.7×10^{-6}	1.1×10^{-6}	4.3×10^{-7}	3.9×10^{-18}
	0.002	9.1×10^{-6}	3.9×10^{-6}	1.7×10^{-6}	7.1×10^{-7}	3.1×10^{-17}
	0.003	1.1×10^{-5}	4.9×10^{-6}	2.2×10^{-6}	9.6×10^{-7}	1.0×10^{-16}
	0.004	1.2×10^{-5}	5.8×10^{-6}	2.6×10^{-6}	1.1×10^{-6}	2.5×10^{-16}
	0.005	1.3×10^{-5}	6.5×10^{-6}	3.0×10^{-6}	1.3×10^{-6}	4.9×10^{-16}

the CTFCWBKES. Additionally, the exact solutions and the absolute errors between the CEADM solutions and the exact solutions are also illustrated in the same figure. Figure 3 depicts the two-dimensional graphs of the solutions $u(x, t)$ and $w(x, t)$ of the CTFCWBKES, obtained using the Cq-HATM, for various alpha values. Figure 4 illustrates the two-dimensional graphical representations of the solutions $u(x, t)$ and $w(x, t)$ of the CTFCWBKES, which have been derived by the utilization of the CEADM, while considering different values of α . The numerical solutions of $u(x, t)$ and $w(x, t)$ found using Cq-HATM for various values of $\alpha = 0.47$, $\alpha = 0.57$, $\alpha = 0.67$, $\alpha = 0.77$, and $\alpha = 1$ for CTFCWBKES are presented in Tables 1-2. Besides, the numerical solutions of $u(x, t)$ and $w(x, t)$ found using CEADM for various values of $\alpha = 0.47$, $\alpha = 0.57$, $\alpha = 0.67$, $\alpha = 0.77$, and $\alpha = 1$ for CTFCWBKES are presented in Tables 3-4.

6 Conclusion

In this study, CTFCWBKES has been examined by new two numerical methods, namely, Cq-HATM and CEADM. The reliability of these new methods has been confirmed by numerical results. By comparing Cq-HATM to existing numerical algorithms, it is seen that be to reduce the amount of work and evaluation time while maintaining a high degree of accuracy, according to the results obtained. Based on the results obtained, it is clear that using CEADM reduces the workload and evaluation time in comparison to other numerical algorithms while maintaining a high degree of accuracy in the results. When the solutions obtained with the two methods are compared with the exact solutions, it is observed that very small errors are obtained. Therefore, it is seen that both of these methods are effective. The validity of the two methods proposed in this paper is demonstrated and the results show that they are effective. These methods can be used to obtain new numerical solutions of fractional order partial differential equations. Conformable differentiation can easily solve partial differential equations numerically. It has been seen that other integral transformations can also be applied to similar methods. It has been seen that methods with absolute errors can also be applied to unsolvable partial differential equations.

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Author contributions

Mrs. Alkan and Phd. Anac prepared figures and proposed method. Mrs. Alkan and Mr. Avit wrote the main manuscript. All authors reviewed the manuscript.

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Data availability

No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate

Not applicable.

Competing interests

The authors declare no competing interests.

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